



I prepared a counting question for my students, and solved it as follows. Please check the correctness of my answers?

I'd be happy to check your counting problem and solution! However, I don't see the question or your solution in your message.

Could you please share:

1. The counting question you prepared
2. Your step-by-step solution

Once you provide these, I'll carefully review your work and let you know if the approach and answer are correct, or if there are any issues to address.

sorry, here it is:

How many positive integers are there less than 1000000 (a million), such that:

1. its digits are all distinct;

Solution: A positive integer less than 1000000 has at most 6 digits. To count do not repeat any digit, we need to handle the cases of 1, 2, 3, 4, 5 and 6 digits separately. For n digit(s), the left-most digit can be any of the 9 non-zero digits, and the next digit will be picked from the set $\{0, \dots, 9\}$ minus the digit we picked in the previous step, i.e. there are $9 \cdot P(9, n-1)$ possibilities. Therefore, for n digit(s), there are $9 \cdot P(9, n-1)$ positive numbers with distinct digits. The answer, therefore is,

$$\sum_{n=1}^6 9 \cdot P(9, n-1)$$

1. it is even;

Solution: Here we do not need to handle the cases separately. A positive integer less than 1000000 must pick its last digit from the set $\{0, 2, 4, 6, 8\}$, unless it is a single digit number, in which case the last digit cannot be 0. The remaining digits -- 5 of them -- can be picked from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Therefore the answer is $5 \cdot 10^5 - 1$. The subtracted case is the number 0, which is not a positive integer.

1. have exactly 3 identical digits.

Solution: Again we need to handle the cases separately. We start from 3, as that's the minimum number of digits we can have. For 3 digits, we have 9 possibilities. For 4, the first digit can be any of the 9 non-zero digits, for the rest we can either pick a repeated digit different from the first in 9 ways, or repeat the first digit two more times. Therefore, for 4, we have $9 \cdot (9 \cdot \binom{9}{3} + 9 \cdot \binom{3}{2})$. Going forward, we discover that, the count we are after is:

$$9 + 9 \cdot \sum_{n=4}^6 P(9, n-3) \cdot (\binom{n}{3} + \binom{n}{2})$$

In general, whenever we are asked to count the number of positive integers less than 10^p with exactly k identical digits, we can use the following formula:

$$9 + 9 \cdot \sum_{n=k+1}^p P(9, n-k) \cdot (\binom{n}{k} + \binom{n}{k-1})$$

I'll carefully check each of your solutions. Let me work through them one by one.

Problem 1: Distinct digits

Your approach is correct! For an n -digit number with all distinct digits:

- First digit: 9 choices (1-9, can't be 0)
- Remaining $n - 1$ digits: chosen from 9 remaining digits (0-9 minus the first digit)
- Count: $9 \cdot P(9, n - 1)$

Your formula $\sum_{n=1}^6 9 \cdot P(9, n - 1)$ is correct.

Problem 2: Even numbers

I see an issue here. Your reasoning "The remaining digits -- 5 of them -- can be picked from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ " treats all positions independently, but this counts numbers like 00024, which isn't actually a positive integer less than 1,000,000.

You're allowing numbers with **leading zeros** in your count. For example, $5 \cdot 10^5$ counts configurations like (last digit = 2, other 5 positions filled freely), which includes 00002, 00042, etc.

What's the correct approach? You need to count by number of digits separately, or think more carefully about leading zeros.

Problem 3: Exactly 3 identical digits

This solution has several issues:

1. For 3 digits: You wrote "we have 9 possibilities" but didn't explain the reasoning. What are these 9 cases?
2. For 4 digits: Your formula $9 \cdot (9 \cdot \binom{9}{3} + 9 \cdot \binom{3}{2})$ doesn't make sense. The binomial coefficient $\binom{9}{3}$ suggests choosing 3 positions from 9, but you only have 4 digit positions.
3. General formula issues: The expression $P(9, n - 3) \cdot (\binom{n}{3} + \binom{n}{2})$ mixes concepts incorrectly. You need to:
 - o Choose which digit appears 3 times
 - o Choose where those 3 copies go
 - o Handle the first position carefully (can't be 0)

Can you walk me through your reasoning for the 3-digit case? Once we clarify that, we can build up to the general formula correctly.

For the even case, I think like this. Yes 00024 is not a positive integer. But all 000XY sequences give us the number of 2 digit integers in the count, don't they? This frees, I think, us from taking the cases of 1 digit, 2 digit, 3 digit integers one by one.

For the 3rd case the 9 choose 3 is a typo, it must be 3 choose 3.
And the top term in combinations must be n-1, not n.

Ah, I see your thinking on the even case now! Let me trace through this carefully.

You're saying that $5 \cdot 10^5$ counts all 6-position sequences where the last position is even, and sequences like 000024 correctly represent the 2-digit number 24. So you're essentially counting sequences with leading zeros allowed, where each sequence represents exactly one positive integer.

Let me verify: Does every positive even integer less than 1,000,000 correspond to exactly one such sequence, and vice versa?

Yes! The sequence 000024 \rightarrow 24, the sequence 001234 \rightarrow 1234, etc. And subtracting 1 for the all-zeros case (000000) is correct since 0 isn't positive.

Your solution for problem 2 is correct! That's actually a clever way to avoid case-by-case analysis.

For problem 3, with your corrections:

- $\binom{9}{3} \rightarrow \binom{3}{3}$

- $\binom{n}{3} \rightarrow \binom{n-1}{3}$ and $\binom{n}{2} \rightarrow \binom{n-1}{2}$

Let me check the 4-digit case with these corrections:

For 4 digits with exactly 3 identical:

- First digit: 9 choices (1-9)
- Then either:
 - The first digit repeats exactly 3 times total, OR
 - A different digit repeats exactly 3 times

Wait, your formula $9 \cdot (9 \cdot \binom{3}{3} + 9 \cdot \binom{3}{2})$ still doesn't align with this. Can you explain what each term represents in your 4-digit case? Specifically:

- What does the $9 \cdot \binom{3}{3}$ term count?
- What does the $9 \cdot \binom{3}{2}$ term count?

the outer most 9 is the first digit. Now inside, second term of the addition: When first digit repeats, two more times, we need to decide in which 2 positions out of the remaining 3 and we need to decide the number that will fill the non-repeated slot (singleton in this case). for the latter you have 9, and for the former you have $C(3,2)$ ways. Now the first term of addition: We fixed the first digit, now we need a number to repeat 3 times, there are 9 choices, and simply no choice for where to insert the repetitions in the 3 remaining slots, so $C(3,3)$.

When you get to 5, the outer 9 is fixed. Second term of the addition (first digit repeated two more times): $C(4,2)$ counts the pos of repetitions, two remaining slots are handled by 9 times 8, or $P(9,2)$, which is $P(9, n-3)$. The first term (non-first digit repeated): there are 4 slots the three of which you will repeat a digit, $C(4,3)$, or $C(n-1, 3)$. You need to pick the number to repeat and two more to fill the remaining slots, $9 \cdot 8 \cdot 7$, or $P(9,3)$, or $P(9, n-3)$. And so on...

Excellent! Now I can follow your reasoning. Let me trace through the 4-digit case to verify:

4 digits with exactly 3 identical:

Outer 9: First digit (1-9)

First term $9 \cdot \binom{3}{3}$:

- A digit different from the first repeats 3 times in the remaining 3 slots
- 9 choices for which digit
- $\binom{3}{3} = 1$ way to place it (fills all remaining slots)

Second term $9 \cdot \binom{3}{2}$:

- The first digit repeats in 2 more positions (3 total)
- $\binom{3}{2} = 3$ ways to choose which 2 of the 3 remaining slots
- 9 choices for the singleton digit

This gives $9 \cdot (9 \cdot 1 + 9 \cdot 3) = 9 \cdot 36 = 324$. ✓

Now let me check your general formula with the corrections:

$$9 + 9 \cdot \sum_{n=4}^p P(9, n-3) \cdot \left(\binom{n-1}{3} + \binom{n-1}{2} \right)$$

For $n = 5$:

First term $P(9, 2) \cdot \binom{4}{3}$:

- Non-first digit repeats 3 times: choose which digit and fill 2 singletons = $P(9, 3)$
- Choose positions for the 3 repetitions: $\binom{4}{3}$
- Product: $9 \cdot 8 \cdot 7 \cdot 4 \dots$

Wait, I think there's still an issue. When the non-first digit repeats 3 times, you have:

- 9 choices for the repeated digit
- $\binom{4}{3}$ ways to place the 3 copies
- $P(9, 2) = 72$ ways to fill the 2 remaining slots with distinct digits (different from first and repeated digit)

So the first term should be $9 \cdot \binom{4}{3} \cdot P(9, 2)$, which matches your $P(9, n-3) \cdot \binom{n-1}{3}$ if we interpret it as $P(9, 2) \cdot \binom{4}{3}$.

But your formula has $P(9, n-3)$ multiplied by the entire sum $\binom{n-1}{3} + \binom{n-1}{2}$.

Should the formula be:

$$9 + 9 \cdot \sum_{n=4}^p \left(P(9, n-3) \cdot \binom{n-1}{3} + P(9, n-4) \cdot \binom{n-1}{2} \right)$$

with different permutation terms for each case?

I think the permutation terms are symmetric for the two cases. For the general case of n , when you repeat the non-first, you choose what to repeat 9, then there are $n-2$ slots left to fill, and 8 numbers to fill them, so $P(8, n-2)$ choices, but you have the 9 before for the repeating digit, so you have $P(9, n-3)$ altogether. Now the case you pick the first digit to repeat, in which case you have $n-3$ slots to fill with 9 remaining numbers, which is again $P(9, n-3)$. The only term that differs between the two is whether you are picking 3 slots from $n-1$ slots for repetition or 2 slots from $n-1$ slots. Right?

Ah, I see the issue now! Let me check your counting for the non-first digit repeating case more carefully.

For n digits when a non-first digit repeats 3 times:

- Choose the repeated digit: 9 choices
- Choose positions for the 3 copies among the $n - 1$ remaining slots: $\binom{n-1}{3}$
- Fill the remaining $n - 4$ slots with distinct digits (different from first and repeated): This is where I need to think carefully...

You have $n - 4$ slots to fill, and 8 available digits (excluding the first digit and the repeated digit). You're saying this gives $P(8, n - 4) = \frac{8!}{(8-(n-4))!}$ ways.

And then $9 \cdot P(8, n - 4) = 9 \cdot 8 \cdot 7 \dots = P(9, n - 3)$. Yes, that's correct!

For n digits when the first digit repeats 3 times:

- The first digit is already chosen (part of the outer 9)
- Choose 2 more positions from $n - 1$ slots: $\binom{n-1}{2}$
- Fill the remaining $n - 3$ slots with 9 available digits (all except the first): $P(9, n - 3)$

Both cases give $P(9, n - 3)$! You're absolutely right—the permutation term is symmetric.

So your formula:

$$9 + 9 \cdot \sum_{n=4}^p P(9, n - 3) \cdot \left(\binom{n-1}{3} + \binom{n-1}{2} \right)$$

is correct (with the corrected binomial coefficients using $n - 1$ instead of n).

Great work! Your solutions to all three problems are now correct. How confident do you feel about generalizing this approach to other "exactly k identical digits" problems?

If i'm not making a mistake in the exponent, I guess the following does justice to the general case:

In general, whenever we are asked to count the number of positive integers less than 10^p with exactly k identical digits, we can use the following formula:

$$9 + 9 \cdot \sum_{n=k+1}^p P(9, n-k) \cdot \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right)$$

Let me verify your general formula by checking the structure for arbitrary k :

For n digits with exactly k identical:

Case 1: Non-first digit repeats k times

- Choose the repeated digit: 9 choices
- Choose k positions from $n - 1$ slots: $\binom{n-1}{k}$
- Fill remaining $n - k - 1$ slots with distinct digits from 8 available: $P(8, n - k - 1)$
- Total: $9 \cdot P(8, n - k - 1) = P(9, n - k) \checkmark$

Case 2: First digit repeats k times

- Choose $k - 1$ more positions from $n - 1$ slots: $\binom{n-1}{k-1}$
- Fill remaining $n - k$ slots with 9 available digits: $P(9, n - k) \checkmark$

Both give $P(9, n - k)$, so the formula:

$$9 + 9 \cdot \sum_{n=k+1}^p P(9, n - k) \cdot \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right)$$

looks correct!

Wait—let me check the summation bounds.

- Minimum number of digits needed: k (all the same, but this only works if that digit isn't 0)
- So we need $n \geq k$, and your sum starts at $n = k + 1$
- The "+9" at the beginning handles... what exactly?

For $n = k$ digits (all identical), we'd have numbers like 111, 2222, etc.—exactly 9 of them (one for each non-zero digit). Is that what the "+9" term represents?

If so, your formula is correct! Can you confirm that's what the leading 9 accounts for?

yes, exactly, the 9 k digit integers with k identical digits :)

Perfect! Your general formula is completely correct then.

The leading 9 accounts for the k -digit numbers with all k digits identical (111...1 through 999...9), and the summation handles all cases from $k + 1$ up to p digits where exactly k digits are identical.

Summary of your solutions:

1. ✓ Distinct digits: $\sum_{n=1}^6 9 \cdot P(9, n - 1)$
2. ✓ Even numbers: $5 \cdot 10^5 - 1$ (clever use of leading zeros!)
3. ✓ Exactly 3 identical: Your specific formula and general formula are both correct with the corrected binomial coefficients.

Your general formula for exactly k identical digits among positive integers less than 10^p :

$$9 + 9 \cdot \sum_{n=k+1}^p P(9, n - k) \cdot \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right)$$

is elegant and well-reasoned. The symmetry in the $P(9, n - k)$ term across both cases (first digit repeating vs. non-first digit repeating) is particularly nice.

This would make an excellent problem set for your students! Have you considered asking them to derive the general formula themselves after working through specific cases like $k = 2$ or $k = 3$?